



Global s -solvability and global s -hypoellipticity for certain perturbations of zero order of systems of constant real vector fields

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Abstract

The main purpose of this paper is to show, in the two-dimensional torus, a necessary and sufficient condition in order to certain perturbations of zero order of a system of constant real vector fields to be globally s -solvable. We are also interested in studying its global s -hypoellipticity. We present connections between these global concepts and a priori estimates. We also present two applications of our results for systems of operators with variable coefficients.

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1. Introduction

Global solvability and global hypoellipticity give rise to interesting open questions in linear partial differential equations. When the underlying space is the torus these global concepts are often connected to Diophantine phenomena, and number theory enters into the picture. Global solvability for linear partial differential operators on the torus $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$ has been

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studied by many authors, including Albanese and Zanghirati [1], Albanese and Popivanov [2], Bergamasco, Cordaro and Petronilho [6,7], Bergamasco, Nunes, and Zani [10], Bergamasco and Petronilho [11], Cardoso and Hounie [14], Gramchev, Popivanov and Yoshino [22–24], Gramchev and Yoshino [25], Hounie [35], and Petronilho [36–39].

For some interesting results on global s -hypoellipticity we refer the reader to the following papers as well as the references therein: Amano [3], Bergamasco [5], Bergamasco, Nunes, and Zani [9], Bove and Treves [13], Bergamasco and Zani [12], Christ [16], Cordaro and Himonas [17,18], Fujiwara and Omori [21], Greenfield and Wallach [26], Helffer [27], Himonas [28,29], Himonas and Petronilho [30–33], Himonas, Petronilho and dos Santos [34], Tartakoff [42].

There are few results regarding global solvability or global hypoellipticity for operators which are given as perturbations of lower order terms (including pseudodifferential operators) of a globally solvable or globally hypoelliptic operator, respectively. We refer the reader to Bergamasco [4], Dickinson, Gramchev and Yoshino [20], Gramchev, Popivanov and Yoshino [24], Gramchev and Yoshino [25], and Ruzhansky and Turunen [41].

The main purpose of this paper is to answer the following question: if a system of constant real vector fields is globally s -solvable on the torus \mathbb{T}^2 , are its perturbations of zero order globally s -solvable?

As it is well known there is a connection between the concepts of global s -solvability and of global s -hypoellipticity, therefore we will also study this connection.

For this, we start by studying these concepts for systems of constant real vector fields and we point out that, in certain subspaces of the Gevrey spaces, global s -solvability and global s -hypoellipticity are both equivalent to the condition that their coefficients satisfy a Diophantine condition (see Section 2).

In Section 3 we consider a system of constant real vector fields in \mathbb{T}^2 and we analyze the global s -solvability and the global s -hypoellipticity for certain perturbations of zero order.

In this study it is fundamental the role of a certain subgroup, $\Gamma \subset \mathbb{Z}^2$, associated to the system as in [20]. In our case, we have $\Gamma = \{0\}$ (the non-resonant case) or its dimension is one (resonant line). The case when the dimension of Γ is one needs a more careful analysis. We use a suitable change of variables in order to reduce the original system to a simpler one. More precisely, we start with a system of operators \mathcal{P} given by

$$P_j = \sum_{k=1}^2 a_{jk} D_{x_k} + b_j(x_1, x_2), \quad j = 1, \dots, m,$$

where $a_{jk} \in \mathbb{R}$, $b_j \in G^s(\mathbb{T}^2; \mathbb{C})$, and we associate to it the following important subgroup of \mathbb{Z}^2 : $\Gamma(\mathcal{P}) = \{\xi \in \mathbb{Z}^2: \langle \omega^j, \xi \rangle = 0, \forall j = 1, \dots, m\}$, where $\omega^j = (a_{j1}, a_{j2})$.

We introduce a suitable change of variables such that in the new variables, $y = (y_1, y_2) \in \mathbb{T}^2$, the system \mathcal{P} becomes the system \mathcal{R} given by

$$R_j = c_j D_{y_2} + d_j(y_1),$$

where $d_j \in G^s(\mathbb{T}; \mathbb{C})$, $j = 1, \dots, m$, and $c_j \neq 0$ for some $j \in \{1, \dots, m\}$. Thus we can use the partial Fourier transform in order to study its global s -solvability and its global s -hypoellipticity.

In this study, the most interesting case is when there exists $j_0 \in \{1, \dots, m\}$ such that d_{j_0} is non-constant and $\{-c_j \eta: \eta \in \mathbb{Z}\} \cap \mathcal{R}(d_j) \neq \emptyset$, $j = 1, \dots, m$, where $\mathcal{R}(d_j)$ is the range of the function d_j .

Assuming that

$$\{-c_1\eta: \eta \in \mathbb{Z}\} \cap \mathbf{R}(d_1) = \{-c_1\eta_1, \dots, -c_1\eta_N\}$$

we define the following functions:

$$r_{jk}(y_1) = d_j(y_1) + c_j\eta_k, \quad D_k(y_1) = \sum_{\ell=1}^m |r_{\ell k}(y_1)|^2, \quad j = 1, \dots, m, \quad k = 1, \dots, N.$$

In this situation we prove that the system \mathcal{R} is globally s -solvable if and only if

$$\mathcal{F}_k = \bigcap_{j=1}^m \{y_1 \in \mathbb{T}: r_{jk} \text{ is flat at } y_1\} = \emptyset$$

for $k = 1, \dots, N$.

We also prove that the system \mathcal{R} is globally s -hypoelliptic on \mathbb{T}^2 if and only if $D_k(y_1) \neq 0$ for all $y_1 \in \mathbb{T}$, $k = 1, \dots, N$.

We notice that if $y_1 \in \mathcal{F}_k$ then $D_k(y_1) = 0$ for all k and, therefore, if the system \mathcal{R} is globally s -hypoelliptic on \mathbb{T}^2 then one must have $\mathcal{F}_k = \emptyset$, $k = 1, \dots, N$. Thus, the system \mathcal{R} is globally s -solvable. Of course one can have $D_{k_0}(y_1) = 0$ for some $y_1 \in \mathbb{T}$ and some $k_0 \in \{1, \dots, N\}$ with $\mathcal{F}_k = \emptyset$ for any $k \in \{1, \dots, N\}$, which means that the system \mathcal{R} cannot be globally s -hypoelliptic on \mathbb{T}^2 , but it is globally s -solvable.

For the other situations, in the case when the dimension of Γ is one, we prove that the system under study is always globally s -solvable while it may or may be not globally s -hypoelliptic on \mathbb{T}^2 .

In the case that $\Gamma = \{0\}$ we have that the system \mathcal{R} is globally s -solvable if and only if it is globally s -hypoelliptic.

It follows from the comments above that the connection between these global concepts depends on the set Γ .

In the end of Section 3 we present the statement of our main result in the original variables since first we had proved it using a change of variables.

In Section 4 we present two applications of our results. To be more precise, in the first one we consider perturbations of zero order of a class of systems of real vector fields with variable coefficients and we present a necessary and sufficient condition for it to be globally ∞ -solvable. In the second one we consider a system of vector fields with variable coefficients such that it can be reduced simultaneously to a system of constant real vector fields and, therefore, its global s -solvability can be studied by means of Theorem 2.6.

Finally, in the last section we discuss some relations between global properties and a priori estimates in the Sobolev spaces $H^s(\mathbb{T}^2)$.

2. Global s -solvability and global s -hypoellipticity for systems of constant real vector fields

Our goal in this section is to present a necessary and sufficient condition for a system of constant real vector fields on \mathbb{T}^2 to be either globally s -solvable or globally (s, Γ^c) -hypoelliptic on \mathbb{T}^2 (see Definitions 2.1 and 2.3, respectively).

In order to state our result we need to recall some definitions and set some notations.

Let $s \geq 1$. We say that a function $f(x) \in C^\infty(\mathbb{T}^2)$ is in the Gevrey class $G^s(\mathbb{T}^2)$ if there exists a constant $C > 0$ such that $|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s$, for all $\alpha \in \mathbb{Z}_+^2$, $x \in \mathbb{T}^2$. In particular, $G^1(\mathbb{T}^2)$ is the space of all periodic real analytic functions, denoted by $C^\omega(\mathbb{T}^2)$. We denote by $\mathcal{D}'_s(\mathbb{T}^2)$ the dual space of $G^s(\mathbb{T}^2)$ and its elements are called ultradistributions of order s . One can prove that $u \in \mathcal{D}'_s(\mathbb{T}^2)$ is in $G^s(\mathbb{T}^2)$ if and only if there exist positive constants ε and C such that

$$|\hat{u}(\xi)| \leq C e^{-\varepsilon|\xi|^{1/s}}, \quad \forall \xi \in \mathbb{Z}^2 \setminus \{0\}.$$

We set $G^\infty(\mathbb{T}^2) \doteq C^\infty(\mathbb{T}^2)$ and $\mathcal{D}'_\infty(\mathbb{T}^2) \doteq \mathcal{D}'(\mathbb{T}^2)$.

Next we study the global s -solvability for the following system of constant real vector fields

$$L_j = \sum_{k=1}^2 a_{jk} D_{x_k}, \quad j = 1, \dots, m, \quad (2.1)$$

acting on $G^s(\mathbb{T}^2)$, where $a_{jk} \in \mathbb{R}$, and $D_{x_k} = \frac{1}{i} \partial_{x_k}$, $j = 1, \dots, m$, $k = 1, 2$. We denote the system L_j , $j = 1, \dots, m$, by \mathcal{L} .

It is easy to see that given $f_j \in G^s(\mathbb{T}^2)$ such that there exists $u \in G^s(\mathbb{T}^2)$ satisfying $L_j u = f_j$, $j = 1, \dots, m$, then $L_j f_k = L_k f_j$, $j, k \in \{1, \dots, m\}$, and for all $j \in \{1, \dots, m\}$ one has $\langle w, f_j \rangle = 0$ for all $w \in \bigcap_{\ell=1}^m \ker {}^t L_\ell$.

Thus, in order to study global solvability, we restrict ourselves to Gevrey functions f_1, \dots, f_m satisfying the conditions above. More precisely, we define the following set

$$\mathcal{G}^s(\mathcal{L}) = \left\{ (f_1, \dots, f_m): f_j \in G^s(\mathbb{T}^2), L_k f_j = L_j f_k, j, k \in \{1, \dots, m\} \right. \\ \left. \text{and for } j \in \{1, \dots, m\}, \langle w, f_j \rangle = 0 \text{ for all } w \in \bigcap_{\ell=1}^m \ker {}^t L_\ell \right\}$$

and

Definition 2.1. We say that the system \mathcal{L} is globally s -solvable if for any $(f_1, \dots, f_m) \in \mathcal{G}^s(\mathcal{L})$ there exists $u \in G^s(\mathbb{T}^2)$ such that $L_j u = f_j$, $j = 1, \dots, m$.

Remark 2.2. Note that there exist other concepts of solvability. For instance, one may look for solutions of \mathcal{L} in $\mathcal{D}'_s(\mathbb{T}^2)$ instead of $G^s(\mathbb{T}^2)$. This notion of global solvability corresponds to the usual definition of local solvability (see, e.g., [1,2,6,8]). For references related to Definition 2.1 see, e.g., [7,10,11]. We would like to point out that it may happen that a system has solution in $\mathcal{D}'_s(\mathbb{T}^2)$ but not in $G^s(\mathbb{T}^2)$.

We now associate to the system of vector fields \mathcal{L} given by (2.1) the following vectors

$$\omega^j = (a_{j1}, \dots, a_{jn}), \quad j = 1, \dots, m.$$

We also define the following set that will play an important role in our study. Let

$$\Gamma \doteq \Gamma(\mathcal{L}) \doteq \{\xi \in \mathbb{Z}^2: \langle \omega^j, \xi \rangle = 0, j = 1, \dots, m\}, \quad \Gamma^c = \mathbb{Z}^2 \setminus \Gamma.$$

Note that the set Γ is either the origin (non-resonant case) or a resonant line.

Let

$$G^s(\mathbb{T}^2) = G^s_\Gamma(\mathbb{T}^2) \oplus G^s_{\Gamma^c}(\mathbb{T}^2),$$

where

$$G^s_\Gamma(\mathbb{T}^2) = \{f \in G^s(\mathbb{T}^2): \hat{f}(\xi) = 0, \xi \in \Gamma^c\}$$

and

$$G^s_{\Gamma^c}(\mathbb{T}^2) = \{f \in G^s(\mathbb{T}^2): \hat{f}(\xi) = 0, \xi \in \Gamma\}.$$

For $f \in G^s(\mathbb{T}^2)$ we write $f = f_\Gamma + f_{\Gamma^c}$, with $f_\Gamma \in G^s_\Gamma(\mathbb{T}^2)$ and $f_{\Gamma^c} \in G^s_{\Gamma^c}(\mathbb{T}^2)$.

In the same fashion we set

$$\mathcal{D}'_s(\mathbb{T}^2) = \mathcal{D}'_{s,\Gamma}(\mathbb{T}^2) \oplus \mathcal{D}'_{s,\Gamma^c}(\mathbb{T}^2).$$

Now we may recall the following definition of global (s, Γ^c) -hypoellipticity:

Definition 2.3. (See, e.g., [6,39].) We say that the system \mathcal{L} is globally (s, Γ^c) -hypoelliptic on \mathbb{T}^2 if the conditions $u \in \mathcal{D}'_{s,\Gamma^c}(\mathbb{T}^2)$ and $L_j u \in G^s_{\Gamma^c}(\mathbb{T}^2)$, $j = 1, \dots, m$, imply that $u \in G^s_{\Gamma^c}(\mathbb{T}^2)$.

Definition 2.4. (See [29].) Let $v_1, \dots, v_m \in \mathbb{R}^2$ and $1 \leq s \leq \infty$. We say that the vectors v_1, \dots, v_m are not simultaneously approximable with exponent s with respect to Γ^c

- (i) if for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for each $\xi \in \Gamma^c$ there exists $j \in \{1, \dots, m\}$ such that

$$|\langle v_j, \xi \rangle| \geq C_\varepsilon e^{-\varepsilon|\xi|^{1/s}},$$

if $1 \leq s < \infty$,

- (ii) if there exist $C > 0$ and $K > 0$ such that for each $\xi \in \Gamma^c$ there exists $j \in \{1, \dots, m\}$ such that

$$|\langle v_j, \xi \rangle| \geq \frac{C}{|\xi|^K},$$

if $s = \infty$.

In this case we say that the vectors v_1, \dots, v_m are not SA_s ; otherwise, we say that they are SA_s .

It follows from the definition of global s -solvability that the study of the kernel of the transpose of the vector fields L_j will be useful in the sequence. By using Fourier series one can easily prove the following

Lemma 2.5. *Let $w \in \mathcal{D}'_s(\mathbb{T}^2)$. Then $w \in \bigcap_{j=1}^m \ker {}^t L_j$ if and only if $w(x) = \sum_{\xi \in \Gamma} \hat{w}(\xi) e^{i\langle x, \xi \rangle}$.*

We are now in position to state a result, whose proof will be omitted, that will be useful in the sequence of this work.

Theorem 2.6. *The following conditions are equivalent:*

- (I) *the vectors $\omega^1, \dots, \omega^m$ are not SA_s ;*
- (II) *for each $(f_1, \dots, f_m) \in \mathcal{G}^s(\mathcal{L})$ there exists a unique $u \in G^s_{\Gamma^c}(\mathbb{T}^2)$ such that $L_j u = f_j$, $j = 1, \dots, m$;*
- (III) *the system \mathcal{L} is globally (s, Γ^c) -hypoelliptic on \mathbb{T}^2 .*

Observe that condition (II) is equivalent to Definition 2.1.

3. Perturbation

As before, let $1 \leq s \leq \infty$. In this section we take \mathbb{T}^2 as our underlying space. As in (2.1), we consider the system \mathcal{L} given by

$$L_j = \sum_{k=1}^2 a_{jk} D_{x_k}, \quad j = 1, \dots, m,$$

where $x = (x_1, x_2) \in \mathbb{T}^2$ and $a_{jk} \in \mathbb{R}$, $j = 1, \dots, m$, $k = 1, 2$.

We now are interested in the following perturbed system \mathcal{P} given by

$$P_j = L_j + b_j(x),$$

where $b_j \in G^s(\mathbb{T}^2; \mathbb{C})$.

The non-resonant case ($\Gamma = \{0\}$) is quite simple and we mention that a standard computation shows that the system \mathcal{P} is globally s -solvable if and only if it is globally s -hypoelliptic on \mathbb{T}^2 . For the rest of this section we analyze only when Γ has dimension one.

3.1. Global s -solvability

In this subsection we are concerned with the following question: if the system \mathcal{L} is globally s -solvable on \mathbb{T}^2 , are its perturbations $P_j = L_j + b_j(x)$, with $b_j \in G^s(\mathbb{T}^2)$, likewise globally s -solvable?

We first analyze the case $m = 1$.

Let $L = a_1 D_{x_1} + a_2 D_{x_2}$, $a_1, a_2 \in \mathbb{R}$, and assume that L is globally s -solvable. Also, let $b(x) \in G^s(\mathbb{T}^2; \mathbb{C})$ be given.

By recalling that for $w \in \mathcal{D}'_s(\mathbb{T}^2)$ we have

$$\langle w, b_{\Gamma^c} \rangle = (2\pi)^2 \sum_{\xi \in \mathbb{Z}^2} \hat{w}(\xi) \widehat{b_{\Gamma^c}}(-\xi),$$

it follows from Lemma 2.5 that

$$\langle w, b_{\Gamma^c} \rangle = 0, \quad \forall w \in \ker {}^t L.$$

Thus, $b_{\Gamma^c} \in \mathcal{G}^s(L)$ and, therefore, the global s -solvability study for $L + b$ is equivalent to the global s -solvability study for $L + b_{\Gamma}$.

Therefore, motivated by the case $m = 1$, from now on, we will assume that the system \mathcal{L} is globally s -solvable, $b_j \in \mathcal{G}^s(\mathbb{T}^2)$, and $(b_{1\Gamma^c}, \dots, b_{m\Gamma^c}) \in \mathcal{G}^s(\mathcal{L})$, that is, $L_k b_{j\Gamma^c} = L_j b_{k\Gamma^c}$, since $\langle w, b_{j\Gamma^c} \rangle = 0$, for all $w \in \bigcap_{\ell=1}^m \ker {}^t L_\ell$, $j = 1, \dots, m$.

Due to the considerations above we can use the same definition of global s -solvability for \mathcal{P} as the one we have used for \mathcal{L} .

Remark 3.1. Thanks to the fact that $(b_{1\Gamma^c}, \dots, b_{m\Gamma^c}) \in \mathcal{G}^s(\mathcal{L})$ and the system \mathcal{L} is globally s -solvable there exists $h \in \mathcal{G}^s(\mathbb{T}^2)$ such that $L_j h = b_{j\Gamma^c}$, $j = 1, \dots, m$. Thus, it is easy to prove that the system \mathcal{P} is globally s -solvable if and only if the system $Q_j = L_j + b_{j\Gamma}(x)$, $j = 1, \dots, m$, is globally s -solvable. For this it suffices to note that $(f_1, \dots, f_m) \in \mathcal{G}^s(\mathcal{P})$ if and only if $(e^h f_1, \dots, e^h f_m) \in \mathcal{G}^s(\mathcal{Q})$ and that $P_j u = f_j \Leftrightarrow Q_j v = e^h f_j \doteq g_j$ where $u = e^{-h} v$ and \mathcal{Q} represents the system Q_j , $j = 1, \dots, m$.

It follows from the remark above that we may study the global s -solvability for the system \mathcal{Q} given by

$$Q_j = L_j + b_{j\Gamma}(x), \quad j = 1, \dots, m, \quad (3.1)$$

instead of \mathcal{P} .

Before we proceed we present an example of the situation when $\dim \Gamma = 1$.

For $x = (x_1, x_2) \in \mathbb{T}^2$ we consider the system \mathcal{L} given by

$$L_j = \frac{p_{j1}}{q_{j1}} D_{x_1} + \frac{p_{j2}}{q_{j2}} D_{x_2}, \quad j = 1, 2,$$

where $(p_{jk}, q_{jk}) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, $j, k = 1, 2$, satisfy

$$\frac{p_{11}p_{22}}{q_{11}q_{22}} = \frac{p_{12}p_{21}}{q_{12}q_{21}}.$$

We may, as we will, assume that $\gcd(p_{12}q_{11}, q_{12}p_{11}) = 1$.

In this example we have

$$\omega^j = \left(\frac{p_{j1}}{q_{j1}}, \frac{p_{j2}}{q_{j2}} \right), \quad j = 1, 2.$$

It is easy to see that in this case the set Γ is given by

$$\Gamma = [(-p_{12}q_{11}, q_{12}p_{11})]$$

and, therefore, $\dim \Gamma = 1$.

We also notice that if $(\xi_1, \xi_2) \notin \Gamma$ then

$$\begin{aligned} |\langle \omega^1, \xi \rangle| &= \left| \frac{p_{11}}{q_{11}} \xi_1 + \frac{p_{12}}{q_{12}} \xi_2 \right| \\ &= \frac{|p_{11}q_{12}\xi_1 + p_{12}q_{11}\xi_2|}{|q_{11}q_{12}|} \\ &\geq \frac{1}{|q_{11}q_{12}|} \doteq C. \end{aligned}$$

By noticing that $1 \geq \frac{1}{|\xi|}$ and $1 \geq e^{-\varepsilon|\xi|^{1/s}}$ for any $\xi \neq 0$ and for any $\varepsilon > 0$ we can conclude that the vectors ω^1, ω^2 are not simultaneously approximable with exponent s with respect to Γ^c .

It follows from Theorem 2.6 that the system \mathcal{L} is globally s -solvable.

Now we are back to the general settings. We recall some useful results about algebra and for more details we refer the reader to [39, Section 4].

Since Γ is a subgroup of \mathbb{Z}^2 and we are assuming that its dimension is one then there is one vector $k^1 = (k_1^1, k_2^1) \in \mathbb{Z}^2 \setminus \{0\}$ such that $\{k^1\}$ is a \mathbb{Z} -basis of Γ . Note that $\gcd(k_1^1, k_2^1) = 1$, since, otherwise, $d \doteq \gcd(k_1^1, k_2^1) \geq 2$ and writing $(k_1^1, k_2^1) = d(a, b)$ we would have $(a, b) \in \Gamma$ but $(a, b) \notin [k^1] \doteq \{\ell k^1, \ell \in \mathbb{Z}\}$. Since $\{k^1\}$ is a \mathbb{Z} -basis of Γ there exists a vector $v^2 = (v_1^2, v_2^2) \in \mathbb{Z}^2 \setminus \{0\}$ such that $\{k^1, v^2\}$ form a \mathbb{Z} -basis of \mathbb{Z}^2 .

We will use the \mathbb{Z} -basis $\{k^1, v^2\}$ of \mathbb{Z}^2 in order to define a change of variables. We consider the matrix M such that its columns coincide with k^1, v^2 . One can prove (see [39]) that M^t induces an isomorphism from the torus \mathbb{T}^2 onto itself.

Thus we may make the following change of variables in \mathbb{T}^2 :

$$y = M^t x.$$

We are going to write the operators Q_j , $j = 1, \dots, m$, in the new variables. For this we note that if

$$v(y) = v(M^t x) = u(x)$$

then

$$\begin{aligned} \sum_{\ell=1}^2 a_{j\ell} \frac{\partial u(x)}{\partial x_\ell} &= \sum_{\ell=1}^2 a_{j\ell} \left(\sum_{k=1}^2 \frac{\partial v(y)}{\partial y_k} (M^t)_{k\ell} \right) \\ &= \sum_{k=1}^2 \left(\sum_{\ell=1}^2 (M^t)_{k\ell} a_{j\ell} \right) \frac{\partial v(y)}{\partial y_k} = \sum_{k=1}^2 c_{jk} \frac{\partial v(y)}{\partial y_k} \end{aligned}$$

with

$$c_{jk} = \sum_{\ell=1}^2 (M^t)_{k\ell} a_{j\ell}.$$

We set

$$\theta^j \doteq (c_{j1}, c_{j2}) = M^t \omega^j.$$

Hence, in the new variables, $y = (y_1, y_2)$ the operators Q_j , $j = 1, \dots, m$, become

$$R_j = c_{j2} D_{y_2} + b_{j\Gamma}((M^t)^{-1}y) \quad \text{in } \mathbb{T}_y^2$$

since

$$c_{j1} = \langle \omega^j, k^1 \rangle = 0, \quad j = 1, \dots, m \text{ and } x = (M^t)^{-1}y.$$

We also notice that there exists $j \in \{1, \dots, m\}$ such that $c_{j2} \neq 0$ since $c_{j2} = \langle \omega^j, v^2 \rangle$ and $v^2 \notin \Gamma$.

Next we show that the function $b_{j\Gamma}((M^t)^{-1}y)$ does not depend neither on the variable y_2 nor on the choice of the vector v^2 of the \mathbb{Z} -basis $\{k^1, v^2\}$.

First of all we recall that

$$b_{j\Gamma}(x) = \sum_{\xi \in \Gamma} \widehat{b_{j\Gamma}}(\xi) e^{i\langle x, \xi \rangle}.$$

Thus,

$$\begin{aligned} b_{j\Gamma}((M^t)^{-1}y) &= \sum_{\xi \in \Gamma} \widehat{b_{j\Gamma}}(\xi) e^{i\langle (M^t)^{-1}y, \xi \rangle} \\ &= \sum_{\xi \in \Gamma} \widehat{b_{j\Gamma}}(\xi) e^{i\langle y, M^{-1}\xi \rangle} \\ &= \sum_{M\eta \in \Gamma} \widehat{b_{j\Gamma}}(M\eta) e^{i\langle y, \eta \rangle}, \end{aligned} \tag{3.2}$$

where $\eta = M^{-1}\xi$.

Next we observe that

$$\begin{aligned} M\eta &= \begin{pmatrix} k_1^1 & v_1^2 \\ k_2^1 & v_2^2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} k_1^1 \eta_1 + v_1^2 \eta_2 \\ k_2^1 \eta_1 + v_2^2 \eta_2 \end{pmatrix} \\ &= \eta_1 \begin{pmatrix} k_1^1 \\ k_2^1 \end{pmatrix} + \eta_2 \begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} = \eta_1 k^1 + \eta_2 v^2. \end{aligned}$$

Thanks to the fact that $M\eta \in \Gamma$ we must have

$$0 = \langle \omega^j, M\eta \rangle = \langle \omega^j, \eta_1 k^1 + \eta_2 v^2 \rangle = \eta_2 \langle \omega^j, v^2 \rangle, \quad \text{for all } j \in \{1, \dots, m\},$$

since $\langle \omega^j, k^1 \rangle = 0$.

Thus we must have $\eta_2 = 0$, since $\langle \omega^j, v^2 \rangle \neq 0$ for some $j \in \{1, \dots, m\}$. Hence $M\eta = \eta_1 k^1$ and it follows from this and from (3.2) that

$$\begin{aligned} b_{j\Gamma}((M^t)^{-1}y) &= \sum_{M\eta \in \Gamma} \widehat{b_{j\Gamma}}(M\eta) e^{i\langle y, \eta \rangle} \\ &= \sum_{\eta_1 \in \mathbb{Z}} \widehat{b_{j\Gamma}}(\eta_1 k^1) e^{iy_1 \eta_1} \doteq d_j(y_1). \end{aligned} \quad (3.3)$$

Summing up, we have proved what we desired.

If $\{\tilde{k}^1, \tilde{v}^2\}$ is another \mathbb{Z} -basis of \mathbb{Z}^2 which has the same properties as the \mathbb{Z} -basis $\{k^1, v^2\}$ then from the observations above we may write

$$\begin{aligned} d_j(y_1) &= b_{j\Gamma}((M^t)^{-1}y) = b_{j\Gamma}(v_2^2 y_1 - k_2^1 y_2, -v_1^2 y_1 + k_1^1 y_2) \\ &= b_{j\Gamma}(v_2^2 y_1, -v_1^2 y_1) = b_{j\Gamma}(\tilde{v}_2^2 y_1, -\tilde{v}_1^2 y_1). \end{aligned}$$

Thus, by setting $c_j = c_{j2}$, we write

$$R_j = c_j D_{y_2} + d_j(y_1), \quad j = 1, \dots, m, \quad (3.4)$$

where $d_j \in G^s(\mathbb{T}; \mathbb{C})$ and $c_j \neq 0$ for some $j = 1, \dots, m$.

Remark 3.2. It is easy to prove that the system \mathcal{Q} is globally s -solvable if and only if the system \mathcal{R} is globally s -solvable.

In order to analyze the global s -solvability for the system \mathcal{R} , we split our study in the following cases:

(I) There exists $j_0 \in \{1, \dots, m\}$ such that

$$\{-c_{j_0} \eta : \eta \in \mathbb{Z}\} \cap \mathbf{R}(d_{j_0}) = \emptyset,$$

where $\mathbf{R}(d_{j_0})$ is the range of function d_{j_0} .

(II) $\{-c_j \eta : \eta \in \mathbb{Z}\} \cap \mathbf{R}(d_j) \neq \emptyset$ for all $j \in \{1, \dots, m\}$.

We split the case (II) in two subcases:

(II-1) d_j is constant for all j . Therefore $d_j(y_1) \equiv -c_j \eta_j$, $j = 1, \dots, m$, where $\eta_j \in \mathbb{Z}$.

(II-2) There exists $j_0 \in \{1, \dots, m\}$ such that d_{j_0} is non-constant.

For the cases (I) and (II-1) the following result holds true:

Lemma 3.3. Suppose one has either (I) or (II-1). Then the system \mathcal{R} is globally s -solvable.

Proof. We will present only the proof of the case when (II-1) holds. Thus, $d_j \equiv -c_j \eta_j$, where $\eta_j \in \mathbb{Z}$ and the system \mathcal{R} is given by

$$R_j = c_j D_{y_2} - c_j \eta_j, \quad j = 1, \dots, m.$$

Let $(f_1, \dots, f_m) \in \mathcal{G}^s(\mathcal{R})$. Using Fourier series we must solve

$$(c_j \eta - c_j \eta_j) \hat{u}(y_1, \eta) = \hat{f}_j(y_1, \eta), \quad \forall y_1 \in \mathbb{T}, \eta \in \mathbb{Z}, j = 1, \dots, m.$$

We recall that we are assuming that there exists $j_0 \in \{1, \dots, m\}$ such that $c_{j_0} \neq 0$. We may suppose that $c_1 \neq 0$.

Let $\{j_1, \dots, j_v\} = \{j \in \{1, \dots, m\} : c_j \neq 0\}$.

Now we split the proof in three cases.

Case 1. $v = 1$.

Therefore $j_1 = 1$. In this case we have $c_j = 0$ for $j \in \{2, \dots, m\}$, i.e., the system has only one equation.

We set

$$\hat{u}(y_1, \eta) = \frac{\hat{f}_1(y_1, \eta)}{c_1(\eta - \eta_1)}, \quad \forall y_1 \in \mathbb{T}, \eta \neq \eta_1,$$

which are in $G^s(\mathbb{T})$ for any $\eta \in \mathbb{Z} \setminus \{\eta_1\}$.

Observe that for each $\tilde{y}_1 \in \mathbb{T}$, $\delta(y_1 - \tilde{y}_1) \otimes e^{-iy_2 \eta_1} \in \bigcap_{\ell=1}^m \ker {}^t R_\ell = \ker {}^t R_1$ and, therefore, it follows from the compatibility conditions that

$$\hat{f}_1(\tilde{y}_1, \eta_1) = 0, \quad \text{for all } \tilde{y}_1 \in \mathbb{T}.$$

Thus we define

$$\hat{u}(y_1, \eta_1) = 0, \quad \text{for all } y_1 \in \mathbb{T}.$$

Now it is easy to prove that

$$u(y_1, y_2) = \sum_{\eta \in \mathbb{Z}} \hat{u}(y_1, \eta) e^{iy_2 \eta}$$

belongs to $G^s(\mathbb{T}^2)$ and $R_j u = f_j$, $j = 1, \dots, m$.

The proof of Case 1 is complete.

Case 2. $v \geq 2$ and $\eta_{j_k} \doteq \xi$ for all $k \in \{1, \dots, v\}$.

The proof of this case is similar to the proof of Case 1.

Case 3. $v \geq 2$ and $\eta_{j_{k_1}} \neq \eta_{j_{k_2}}$ for some $j_{k_1}, j_{k_2} \in \{j_1, \dots, j_v\}$.

We will assume $j_{k_1} = 1$ and $j_{k_2} = 2$. Therefore we have $d_1 = c_1\eta_1$ and $d_2 = c_2\eta_2$ with $\eta_1 \neq \eta_2$ and $c_1 \neq 0, c_2 \neq 0$.

Now we set

$$\hat{u}(y_1, \eta) = \frac{\hat{f}_1(y_1, \eta)}{c_1(\eta - \eta_1)}, \quad \forall y_1 \in \mathbb{T}, \eta \neq \eta_1,$$

and

$$\hat{u}(y_1, \eta_1) = \frac{\hat{f}_2(y_1, \eta_1)}{c_2(\eta_1 - \eta_2)}, \quad \forall y_1 \in \mathbb{T}.$$

Now it is easy to prove that

$$u(y_1, y_2) = \sum_{\eta \in \mathbb{Z}} \hat{u}(y_1, \eta) e^{iy_2\eta}$$

belongs to $G^s(\mathbb{T}^2)$ and $R_j u = f_j, j = 1, \dots, m$.

The proof of Lemma 3.3 is now complete. \square

We now treat the main case, i.e., the case (II-2), that is, there exists $j_0 \in \{1, \dots, m\}$ such that d_{j_0} is non-constant and $\{-c_j\eta: \eta \in \mathbb{Z}\} \cap \mathbf{R}(d_j) \neq \emptyset$, for all $j \in \{1, \dots, m\}$.

For the sake of simplicity, we assume that $j_0 = 1$.

Since d_1 is bounded on \mathbb{T} we can conclude that there exists only a finite number of $\eta \in \mathbb{Z}$ such that $-c_1\eta$ belongs to the range of d_1 , say, η_1, \dots, η_N .

Now we set

$$r_{jk}(y_1) = d_j(y_1) + c_j\eta_k, \quad \forall y_1 \in \mathbb{T}, j = 1, \dots, m, k = 1, \dots, N,$$

and $\mathcal{F}_k = \bigcap_{j=1}^m \{y_1 \in \mathbb{T}: r_{jk} \text{ is flat at } y_1\}$.

Note that $\mathcal{F}_k \cap \mathcal{F}_{k'} = \emptyset$ if $k \neq k'$. Indeed, otherwise there would exist y_1 such that $r_{jk}(y_1) = 0 = r_{jk'}(y_1)$ for all j . Since $c_{j'} \neq 0$ for some $j' \in \{1, \dots, m\}$, we would have $d_{j'}(y_1) = -c_{j'}\eta_k = -c_{j'}\eta_{k'}$, which is a contradiction since $\eta_k \neq \eta_{k'}$.

We now state our main result:

Theorem 3.4. *Suppose that d_1 is non-constant and $\{-c_j\eta: \eta \in \mathbb{Z}\} \cap \mathbf{R}(d_j) \neq \emptyset$, for all $j \in \{1, \dots, m\}$. Then, the system \mathcal{R} is globally s -solvable if and only if $\mathcal{F}_k = \emptyset, k = 1, \dots, N$.*

Before we present the proof of Theorem 3.4, we will need a couple of lemmas.

Lemma 3.5. *Let $\mathcal{V} \doteq \bigcap_{j=1}^m \ker {}^t R_j$ and fix $k \in \{1, \dots, N\}$. We assume that there exists y_1^0 such that $r_{jk}^{(\ell)}(y_1^0) = 0$ for $j = 1, \dots, m, \ell = 0, \dots, p$, for some $p \geq 0$. Then*

$$\delta^{(\ell)}(y_1 - y_1^0) \otimes e^{-iy_2\eta_k} \in \mathcal{V}, \quad \ell = 0, \dots, p.$$

Lemma 3.6. Let $f \in G^s(\mathbb{T})$ be such that $f(0) = f'(0) = 0$. Then the function h given by

$$h(x) = \begin{cases} f(x)/(1 - \cos x), & \text{if } x \notin 2\pi\mathbb{Z}, \\ f''(0), & \text{otherwise} \end{cases}$$

belongs to $G^s(\mathbb{T})$. Furthermore, if f is flat at $x = 0$ then the same is true for h .

Proof. We will prove only the case $1 \leq s < \infty$ since this result has been used in [11] and in [7] in the case $s = \infty$.

It is clear that $h \in C^\infty(\mathbb{T})$.

For each $x \in \mathbb{R}$ define $\lambda(t) = f(tx)$, $t \in [0, 1]$. We have

$$f(x) = \lambda(1) - \lambda(0) = \int_0^1 \lambda'(t) dt = x \int_0^1 f'(tx) dt.$$

For each $x \in \mathbb{R}$ and $t \in [0, 1]$, define $\lambda_1(s) = f'(stx)$, $s \in [0, 1]$. We have

$$f'(tx) = \lambda_1(1) - \lambda_1(0) = \int_0^1 \lambda_1'(s) ds = tx \int_0^1 f''(stx) ds.$$

Hence,

$$f(x) = x^2 \int_0^1 \int_0^1 t f''(stx) ds dt = x^2 h_1(x),$$

where

$$h_1(x) = \int_0^1 \int_0^1 t f''(stx) ds dt.$$

It follows from the definition of h that

$$(1 - \cos x)h(x) = f(x) = x^2 h_1(x), \quad \text{for } x \notin 2\pi\mathbb{Z}.$$

Furthermore, since for $m \in \mathbb{Z} \setminus \{0\}$ we have

$$0 = f(0) = f(2m\pi) = (2m\pi)^2 h_1(2m\pi),$$

we can conclude that $h_1(2m\pi) = 0$ and therefore it follows from the considerations above that

$$(1 - \cos x)h(x) = f(x) = x^2 h_1(x), \quad \text{for all } x.$$

Thus, there exists $\delta_1 \in (0, 2\pi)$ such that $h(x) = \varphi(x)h_1(x)$ for $x \in (-\delta_1, \delta_1)$, where φ is the real analytic function defined in $(-\delta_1, \delta_1)$ by

$$\varphi(x) = \begin{cases} x^2/(1 - \cos x), & x \in (-\delta_1, 0) \cup (0, \delta_1), \\ 2, & x = 0. \end{cases}$$

Since f is in $G^s(\mathbb{T})$ there exists $C_f > 1$ such that

$$|f^{(j)}(x)| \leq C_f^{j+1} (j!)^s, \quad \forall x \in \mathbb{R}, j \geq 0.$$

Hence,

$$\begin{aligned} |h_1^{(j)}(x)| &\leq \sup_{y \in \mathbb{R}} |f^{(j+2)}(y)| = \sup_{y \in [0, 2\pi]} |f^{(j+2)}(y)| \\ &\leq C_f^{j+3} ((j+2)!)^s \leq C_f^{j+3} 2^{3s} 2^{sj} (j!)^s \leq C_0^{j+1} (j!)^s, \end{aligned}$$

where $C_0 = \max\{2^s C_f, 2^{3s} C_f^3\} = 2^{3s} C_f^3$. This shows that $h_1 \in G^s(\mathbb{R})$.

Let $0 < \delta_2 < \delta_1$. Since φ is real analytic in $(-\delta_1, \delta_1)$ there exists $C_1 > 0$ such that

$$|h^{(j)}(x)| = |(\varphi h_1)^{(j)}(x)| \leq C_1^{j+1} (j!)^s, \quad \text{for any } x \in [-\delta_2, \delta_2] \text{ and } j \geq 0.$$

Since h is 2π -periodic we can find in a similar fashion positive constants δ_3 and C_2 such that

$$|h^{(j)}(x)| \leq C_2^{j+1} (j!)^s, \quad \text{for any } x \in [2\pi - \delta_3, 2\pi + \delta_3] \text{ and } j \geq 0.$$

Since $h(x) = f(x)/(1 - \cos x)$ for $x \in (0, 2\pi)$, there exists $C_3 > 0$ such that

$$|h^{(j)}(x)| \leq C_3^{j+1} (j!)^s, \quad \text{for any } x \in [\delta_2, 2\pi - \delta_3] \text{ and } j \geq 0.$$

Therefore, if we set $C = \max\{C_1, C_2, C_3\}$, we obtain

$$|h^{(j)}(x)| \leq C^{j+1} (j!)^s, \quad \text{for any } x \in [0, 2\pi] \text{ and } j \geq 0.$$

Hence $h \in G^s(\mathbb{T})$.

Finally, it is clear that if the origin is a zero of order $k \geq 2$ of f then the origin is a zero of order $k - 2$ of h . In particular, if f is flat at the origin, the same is true for h . \square

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. Necessity. Suppose that there exists $k_0 \in \{1, \dots, N\}$ such that $\mathcal{F}_{k_0} \neq \emptyset$. Therefore, $s > 1$. Let $y_1^0 \in \mathcal{F}_{k_0}$. We will write $\mathcal{F} \doteq \mathcal{F}_{k_0}$.

If $r_{1k_0} \equiv 0$ then $d_1(y_1) \equiv -c_1 \eta_{k_0}$ and this is a contradiction because we are assuming that d_1 is non-constant. Therefore $r_{1k_0} \not\equiv 0$ and, hence, $\mathcal{F} \neq \mathbb{T}^2$. Thus, we have $\partial \mathcal{F} \neq \emptyset$. Let $y_1' \in \partial \mathcal{F}$. Since \mathcal{F} is closed, $y_1' \in \mathcal{F}$.

Since $y_1' \in \partial \mathcal{F}$ we have $B(y_1', 1) \cap \mathcal{F}^c \neq \emptyset$, where $B(y_1', 1)$ means the open interval of center y_1' and radius 1. Thus there exists a point $z_1 \in B(y_1', 1) \cap \mathcal{F}^c$ and $j_1 \in \{1, \dots, m\}$ such that $r_{j_1 k_0}(z_1) \neq 0$ since if we had $r_{j k_0}(y_1) = 0$ for any $j \in \{1, \dots, m\}$ and for any $y_1 \in B(y_1', 1) \cap \mathcal{F}^c$

we would have that for all $j \in \{1, \dots, m\}$, r_{jk_0} would be flat at some point of \mathcal{F}^c , which is a contradiction.

Taking $\delta_1 = \min(d(z_1, \mathcal{F})/2, 1/2)$, where $d(z_1, \mathcal{F})$ means the distance from z_1 to \mathcal{F} , analogously one can prove that there exists $z_2 \in B(y'_1, \delta_1) \cap \mathcal{F}^c$ and $j_2 \in \{1, \dots, m\}$ such that $r_{j_2 k_0}(z_2) \neq 0$. Proceeding with this argument we can find a sequence (z_n) which converges to y'_1 such that for each n , $r_{j_n k_0}(z_n) \neq 0$ for some $j_n \in \{1, \dots, m\}$. Since $\{1, \dots, m\}$ is finite, there exist $p \in \{1, \dots, m\}$ and a subsequence (j_{n_ℓ}) of j_n such that $j_{n_\ell} = p$. Setting $y_{1_\ell} = z_{n_\ell}$ we have $r_{pk_0}(y_{1_\ell}) \neq 0$.

Since $y'_1 \in \mathcal{F}$ it follows from Lemma 3.6 that we may write $r_{jk_0}(y_1) = g(y_1)h_{jk_0}(y_1)$, $j = 1, \dots, m$, with $g(y_1) = 1 - \cos(y_1 - y'_1)$ and $h_{jk_0} \in G^s(\mathbb{T})$. We also have that h_{jk_0} is flat at y'_1 .

Since $r_{pk_0}(y_{1_\ell}) \neq 0$ and $r_{pk_0}(y_1) = g(y_1)h_{pk_0}(y_1)$ we have $h_{pk_0}(y_{1_\ell}) \neq 0$.

Define, for $j = 1, \dots, m$,

$$f_{jk_0}(y_1, y_2) = h_{jk_0}(y_1)e^{iy_2\eta_{k_0}}.$$

Now we are going to show that $(f_{1k_0}, \dots, f_{mk_0}) \in \mathcal{G}^s(\mathcal{R})$. Since $h_{jk_0} \in G^s(\mathbb{T})$ we have $f_{jk_0} \in G^s(\mathbb{T}^2)$ for any $j \in \{1, \dots, m\}$.

We also have

$$\begin{aligned} R_k f_{jk_0}(y_1, y_2) &= (c_k D_{y_2} + d_k(y_1))f_{jk_0}(y_1, y_2) \\ &= (c_k \eta_{k_0} + d_k(y_1))h_{jk_0}(y_1)e^{iy_2\eta_{k_0}} \\ &= r_{kk_0}(y_1)h_{jk_0}(y_1)e^{iy_2\eta_{k_0}} \\ &= g(y_1)h_{kk_0}(y_1)h_{jk_0}(y_1)e^{iy_2\eta_{k_0}} \\ &= g(y_1)h_{jk_0}(y_1)h_{kk_0}(y_1)e^{iy_2\eta_{k_0}} \\ &= r_{jk_0}(y_1)h_{kk_0}(y_1)e^{iy_2\eta_{k_0}} \\ &= (c_j \eta_{k_0} + d_j(y_1))h_{kk_0}(y_1)e^{iy_2\eta_{k_0}} \\ &= (c_j D_{y_2} + d_j(y_1))f_{kk_0}(y_1, y_2) \\ &= R_j f_{kk_0}(y_1, y_2). \end{aligned}$$

Take $v \in \mathcal{V} = \bigcap_{j=1}^m \ker {}^t R_j$. By taking Fourier series in the equations ${}^t R_j v = 0$ we obtain

$$(-c_j \eta + d_j(y_1))\hat{v}(y_1, \eta) = 0, \quad \eta \in \mathbb{Z}, \quad j \in \{1, \dots, m\}. \quad (3.5)$$

Since $-c_1 \eta + d_1(y_1) \neq 0$ for $\eta \notin \{-\eta_1, \dots, -\eta_N\}$ and for all $y_1 \in \mathbb{T}$ it follows from (3.5) that $\hat{v}(y_1, \eta) = 0$ for all $\eta \notin \{-\eta_1, \dots, -\eta_N\}$.

Thus, $v(y_1, y_2) = \sum_{k=1}^N \hat{v}(y_1, -\eta_k)e^{-iy_2\eta_k}$ and $(c_j \eta_k + d_j(y_1))\hat{v}(y_1, -\eta_k) = 0$, $j = 1, \dots, m$ and $k = 1, \dots, N$.

In particular, we have

$$\begin{aligned} g(y_1)h_{jk_0}(y_1)\hat{v}(y_1, -\eta_{k_0}) &= r_{jk_0}(y_1)\hat{v}(y_1, -\eta_{k_0}) \\ &= (c_j \eta_{k_0} + d_j(y_1))\hat{v}(y_1, -\eta_{k_0}) = 0. \end{aligned}$$

It follows that $h_{jk_0}(y_1)\hat{v}(y_1, -\eta_{k_0}) = 0$ in $\mathbb{T} \setminus \{y'_1\}$, $j \in \{1, \dots, m\}$, which implies that

$$h_{jk_0}(y_1)\hat{v}(y_1, -\eta_{k_0}) = A_{jk_0}\delta(y_1 - y'_1) + B_{jk_0}\delta'(y_1 - y'_1), \quad j \in \{1, \dots, m\},$$

for some constants A_{jk_0}, B_{jk_0} .

Since h_{jk_0} is flat at y'_1 for all $j \in \{1, \dots, m\}$, we must have $A_{jk_0} = B_{jk_0} = 0$, i.e., $h_{jk_0}(y_1)\hat{v}(y_1, -\eta_{k_0}) = 0$, $j = 1, \dots, m$.

Therefore,

$$\begin{aligned} \langle v, f_{jk_0} \rangle &= 2\pi \sum_{\eta \in \mathbb{Z}} \langle \hat{v}(y_1, -\eta), \widehat{f_{jk_0}}(y_1, \eta) \rangle \\ &= 2\pi \langle \hat{v}(y_1, -\eta_{k_0}), \widehat{f_{jk_0}}(y_1, \eta_{k_0}) \rangle \\ &= 2\pi \langle \hat{v}(y_1, -\eta_{k_0}), h_{jk_0}(y_1) \rangle \\ &= 2\pi \langle h_{jk_0}(y_1)\hat{v}(y_1, -\eta_{k_0}), 1 \rangle = 0, \end{aligned}$$

that is, $f_{jk_0} \in \mathcal{V}^\perp$, $j = 1, \dots, m$.

Hence, we have proved that $(f_{1k_0}, \dots, f_{mk_0}) \in \mathcal{G}^s(\mathcal{R})$.

Now we are going to prove that there is no $u \in G^s(\mathbb{T}^2)$ such that $R_j u = f_{jk_0}$, $j = 1, \dots, m$.

Let us suppose that there exists $u \in G^s(\mathbb{T}^2)$ satisfying $R_j u = f_{jk_0}$, $j = 1, \dots, m$.

Taking Fourier series with respect to y_2 in the last equations, we obtain

$$(c_j \eta + d_j(y_1))\hat{u}(y_1, \eta) = \widehat{f_{jk_0}}(y_1, \eta), \quad \forall y_1 \in \mathbb{T}, \eta \in \mathbb{Z}, j = 1, \dots, m.$$

In particular, we must have

$$(c_j \eta_{k_0} + d_j(y_1))\hat{u}(y_1, \eta_{k_0}) = \widehat{f_{jk_0}}(y_1, \eta_{k_0}) = h_{jk_0}(y_1), \quad \forall y_1 \in \mathbb{T}, j = 1, \dots, m.$$

By using the definition of the functions r_{jk_0} it follows from the formulas above that

$$r_{jk_0}(y_1)\hat{u}(y_1, \eta_{k_0}) = h_{jk_0}(y_1), \quad \forall y_1 \in \mathbb{T}, j = 1, \dots, m.$$

In particular, we have $g(y_1)h_{pk_0}(y_1)\hat{u}(y_1, \eta_{k_0}) = h_{pk_0}(y_1)$, for all $y_1 \in \mathbb{T}$.

Evaluating the last equations at y_{1_ℓ} and recalling that $h_{pk_0}(y_{1_\ell}) \neq 0$ we obtain $\hat{u}(y_{1_\ell}, \eta_{k_0}) = 1/g(y_{1_\ell})$. Recalling that $g(y_1) = 1 - \cos(y_1 - y'_1)$, we see that $\hat{u}(y_{1_\ell}, \eta_{k_0})$ becomes unbounded as $\ell \rightarrow +\infty$, since y_{1_ℓ} converges to y'_1 , which implies that $u \notin G^s(\mathbb{T}^2)$, a contradiction.

This completes the proof of the necessity.

Sufficiency. Let $(f_1, \dots, f_m) \in \mathcal{G}^s(\mathcal{R})$. We must solve the system

$$R_j u = f_j, \quad j = 1, \dots, m.$$

As before we must solve

$$(c_j \eta + d_j(y_1))\hat{u}(y_1, \eta) = \widehat{f_j}(y_1, \eta), \quad \forall y_1 \in \mathbb{T}, \eta \in \mathbb{Z}, j = 1, \dots, m. \quad (3.6)$$

We recall that we are assuming that

$$\{-c_1 \eta: \eta \in \mathbb{Z}\} \cap \mathbf{R}(d_1) = \{-c_1 \eta_1, \dots, -c_1 \eta_N\}.$$

Thus, we have

$$\hat{u}(y_1, \eta) = \frac{\hat{f}_1(y_1, \eta)}{c_1 \eta + d_1(y_1)}, \quad \forall y_1 \in \mathbb{T}, \quad \eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\},$$

which are in $G^s(\mathbb{T})$ for any $\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}$.

It is easy to see that the compatibility conditions $R_j f_k = R_k f_j$, $j, k \in \{1, \dots, m\}$, imply that

$$(c_j \eta + d_j(y_1))\hat{u}(y_1, \eta) = \hat{f}_j(y_1, \eta), \quad \forall y_1 \in \mathbb{T}, \quad \eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}, \quad j = 1, \dots, m.$$

Thus, it is enough to solve Eq. (3.6) for $y_1 \in \mathbb{T}$ and $\eta \in \{\eta_1, \dots, \eta_N\}$.

For this we fix $k \in \{1, \dots, N\}$ and we solve the equations

$$(c_j \eta_k + d_j(y_1))\hat{u}(y_1, \eta_k) = \hat{f}_j(y_1, \eta_k), \quad \forall y_1 \in \mathbb{T}, \quad j = 1, \dots, m, \quad (3.7)$$

which can be written as

$$r_{jk}(y_1)\hat{u}(y_1, \eta_k) = \hat{f}_j(y_1, \eta_k), \quad \forall y_1 \in \mathbb{T}, \quad j = 1, \dots, m. \quad (3.8)$$

We set

$$D_k(y_1) = \sum_{j=1}^m |r_{jk}(y_1)|^2. \quad (3.9)$$

Since $\mathcal{F}_k = \emptyset$, D_k has only finite order zeros and, therefore, the set $\{y_1 \in \mathbb{T}: D_k(y_1) = 0\}$ is finite.

If $y_1 \in \mathbb{T}$ is such that $D_k(y_1) = 0$ then we denote by $\ell_{jk}(y_1)$ the order of the zero y_1 of the function r_{jk} , $j = 1, \dots, m$, and we denote by $\ell_k \doteq \ell_k(y_1) \doteq \min\{\ell_{jk}(y_1): j = 1, \dots, m\}$.

We define

$$G_k(y_1) = \sum_{j=1}^m \overline{r_{jk}(y_1)} \hat{f}_j(y_1, \eta_k), \quad \forall y_1 \in \mathbb{T},$$

and

$$\hat{u}(y_1, \eta_k) = \begin{cases} \frac{G_k(y_1)}{D_k(y_1)}, & \text{if } D_k(y_1) \neq 0, \\ \frac{G_k^{(2\ell_k)}(y_1)}{D_k^{(2\ell_k)}(y_1)}, & \text{if } D_k(y_1) = 0. \end{cases} \quad (3.10)$$

If $D_k(y_1) \neq 0$ for all $y_1 \in \mathbb{T}$, then $\hat{u}(y_1, \eta_k) = G_k(y_1)/D_k(y_1)$ is clearly in $G^s(\mathbb{T})$.

Now, if $D_k(y_1^0) = 0$ for some $y_1^0 \in \mathbb{T}$, then, in order to prove that $\hat{u}(y_1, \eta_k)$ is in $G^s(\mathbb{T})$, it is enough to show that it is Gevrey in a neighborhood of y_1^0 .

Let z be such that $D_k(z) = 0$, and let I_z be an open interval such that $r_{j_z k}(y_1) \neq 0$ in $I_z \setminus \{z\}$, where $j_z \in \{1, \dots, m\}$ is such that $\ell \doteq \ell_k(z) \doteq \ell_{j_z k}(z)$. Note that $D_k(y_1) \neq 0$ in $I_z \setminus \{z\}$, shrinking I_z if it is necessary.

Since z is a zero of order ℓ of the function r_{jk} then z is a zero of order 2ℓ of the function D_k . Thus, we can write

$$D_k(y_1) = (1 - \cos(y_1 - z))^\ell g_{kz}(y_1), \quad \forall y_1 \in \mathbb{T},$$

where

$$g_{kz}(y_1) = \begin{cases} \frac{D_k(y_1)}{(1 - \cos(y_1 - z))^\ell}, & y_1 \notin z + 2\pi\mathbb{Z}, \\ \frac{D_k^{(2\ell)}(y_1)}{[(1 - \cos(y_1 - z))^\ell]^{(2\ell)}}, & y_1 \in z + 2\pi\mathbb{Z}. \end{cases}$$

Defining

$$\varphi_{kz}(y_1) = \begin{cases} \frac{(y_1 - z)^{2\ell}}{(1 - \cos(y_1 - z))^\ell}, & y_1 \in I_z \setminus \{z\}, \\ \frac{(2\ell)!}{[(1 - \cos(y_1 - z))^\ell]^{(2\ell)}}, & y_1 = z, \end{cases}$$

which is real analytic in I_z since it is a restriction of a holomorphic function, we have $g_{kz}(y_1) = \varphi_{kz}(y_1)D_{kz}(y_1)$ in I_z , where D_{kz} is in $G^s(\mathbb{R})$ and it satisfies

$$D_k(y_1) = (y_1 - z)^{2\ell} D_{kz}(y_1), \quad \text{with } D_{kz}(z) = \frac{D_k^{(2\ell)}(z)}{(2\ell)!}.$$

Proceeding as in the proof of Lemma 3.6 one can prove that D_k is Gevrey in a neighborhood of z .

Now we are going to show that $\hat{u}(y_1, \eta_k)$ given by (3.10) solves the equations $r_{jk}(y_1)\hat{u}(y_1, \eta_k) = \hat{f}_j(y_1, \eta_k)$, $\forall y_1 \in \mathbb{T}$, $j = 1, \dots, m$.

When $D_k(y_1) \neq 0$, then for each $i = 1, \dots, m$, we have

$$\begin{aligned} r_{ik}(y_1)\hat{u}(y_1, \eta_k) &= r_{ik}(y_1) \frac{G_k(y_1)}{D_k(y_1)} \\ &= \frac{\sum_{j=1}^m r_{ik}(y_1) \overline{r_{jk}}(y_1) \hat{f}_j(y_1, \eta_k)}{D_k(y_1)} \\ &= \frac{\sum_{j=1}^m r_{jk}(y_1) \overline{r_{jk}}(y_1) \hat{f}_i(y_1, \eta_k)}{D_k(y_1)} \\ &= \hat{f}_i(y_1, \eta_k) \frac{\sum_{j=1}^m |r_{jk}(y_1)|^2}{D_k(y_1)} \\ &= \hat{f}_i(y_1, \eta_k), \end{aligned}$$

where in the third equality we have used the compatibility conditions $R_i f_j = R_j f_i$.

When $y_1^0 \in \mathbb{T}$ is such that $D_k(y_1^0) = 0$, we have $r_{jk}(y_1^0) = 0$ for all $j = 1, \dots, m$. It follows from Lemma 3.5 that $\delta^{(n)}(y_1 - y_1^0) \otimes e^{-iy_2\eta_k} \in \mathcal{V}$, $n = 0, \dots, \ell_k(y_1^0) - 1$.

Since $(f_1, \dots, f_m) \in \mathcal{G}^s(\mathcal{R})$ we have $f_j \in \mathcal{V}^\perp$ and, therefore,

$$\langle \delta^{(n)}(y_1 - y_1^0) \otimes e^{-iy_2\eta_k}, f_j(y_1, y_2) \rangle = 0 \quad (3.11)$$

for $n = 0, \dots, \ell_k(y_1^0) - 1$ and $j = 1, \dots, m$, which implies that

$$\int_0^{2\pi} \frac{\partial^n f_j}{\partial y_1^n}(y_1^0, y_2) e^{-iy_2 \eta_k} dy_2 = 0$$

for $n = 0, \dots, \ell_k(y_1^0) - 1$ and $j = 1, \dots, m$.

Hence,

$$\frac{\partial^n \widehat{f}_j}{\partial y_1^n}(y_1^0, \eta_k) = 0, \quad \text{for } n = 0, \dots, \ell_k(y_1^0) - 1, \quad j = 1, \dots, m. \quad (3.12)$$

In other words, y_1^0 is a zero of order at least $\ell_k(y_1^0)$ of $\widehat{f}_j(\cdot, \eta_k)$, for any $j = 1, \dots, m$.

It follows from the considerations above that

$$r_{jk}(y_1^0) \widehat{u}(y_1^0, \eta_k) = \widehat{f}_j(y_1^0, \eta_k), \quad j = 1, \dots, m.$$

It also justifies the definition of $\widehat{u}(y_1, \eta_k)$ when $D_k(y_1) = 0$. Thus, we have solved Eq. (3.7).

Now we define

$$u(y_1, y_2) = \sum_{\eta \in \mathbb{Z}} \widehat{u}(y_1, \eta) e^{iy_2 \eta} \quad (3.13)$$

and in order to prove that u belongs to $G^s(\mathbb{T}^2)$ we must show that for each $\eta \in \mathbb{Z}$ we have $\widehat{u}(\cdot, \eta) \in G^s(\mathbb{T})$ and there exist $C > 0$ and $\varepsilon > 0$ such that

$$|\partial_{y_1}^j \widehat{u}(y_1, \eta)| \leq C^{j+1} (j!)^s e^{-\varepsilon |\eta|^{1/s}}, \quad (3.14)$$

$\forall j \geq 0, \forall y_1 \in [0, 2\pi], \forall \eta \in \mathbb{Z} \setminus F$, where F is a finite set.

We will analyze $\widehat{u}(y_1, \eta)$ for $\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}$. We recall that, for $\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}$,

$$\widehat{u}(y_1, \eta) = \frac{\widehat{f}_1(y_1, \eta)}{c_1 \eta + d_1(y_1)}, \quad \forall y_1 \in \mathbb{T}. \quad (3.15)$$

Since $d_1(y_1) \in G^s(\mathbb{T})$ we have $c_1 \eta + d_1(y_1) \in G^s(\mathbb{T})$ for any η . For $\eta \notin \{\eta_1, \dots, \eta_N\}$ we have

$$g(y_1, \eta) \doteq \frac{1}{c_1 \eta + d_1(y_1)} \in G^s(\mathbb{T}). \quad (3.16)$$

Since $d_1 \in G^s(\mathbb{T})$ there exists a positive constant $A > 1$ such that

$$|d_1^{(j)}(y_1)| \leq A^{j+1} (j!)^s, \quad \forall y_1 \in [0, 2\pi], \quad \forall j \geq 0. \quad (3.17)$$

Also there exists $B > 0$ such that

$$|g(y_1, \eta)| \leq B, \quad \forall y_1 \in [0, 2\pi], \quad \forall \eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}. \quad (3.18)$$

It follows from (3.15) and (3.16) that in order to prove (3.14) it suffices to prove the following

Lemma 3.7. *There exists $M > 1$ large enough such that*

$$|\partial_{y_1}^j g(y_1, \eta)| \leq B(AM)^{j+1}(j!)^s, \quad (3.19)$$

$\forall j \geq 0, \forall y_1 \in [0, 2\pi], \forall \eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}$.

Proof. For any $j \in \mathbb{N}$ and $M > 1$, we have

$$\begin{aligned} \frac{1}{M} + \frac{1}{M^2} + \dots + \frac{1}{M^j} &= \frac{1}{M} + \frac{1}{M^2} + \dots + \frac{1}{M^j} + 1 - 1 \\ &\leq \sum_{k=0}^{\infty} \left(\frac{1}{M}\right)^k - 1 \\ &= \frac{1}{1 - \frac{1}{M}} - 1 = \frac{1}{M-1}. \end{aligned}$$

Thus there exists $M > 1$ large enough such that

$$BA \left(\frac{1}{M} + \frac{1}{M^2} + \dots + \frac{1}{M^j} \right) \leq 1, \quad \forall j \in \mathbb{N}. \quad (3.20)$$

We may also take M large enough to have $AM > 1$. We will prove (3.19) by induction on j . It follows from (3.18) that for $j = 0$ we have

$$|g(y_1, \eta)| \leq B \leq B(AM), \quad \forall y_1 \in [0, 2\pi], \forall \eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}.$$

We now assume that (3.19) holds true for any $\ell \leq j-1$ and we will prove that it is also true for j .

Since $(c_1\eta + d_1(y_1))g(y_1, \eta) = 1$ it follows that

$$\sum_{k=0}^j \binom{j}{k} \partial_{y_1}^k (c_1\eta + d_1(y_1)) \partial_{y_1}^{j-k} g(y_1, \eta) = 0.$$

Thus

$$\partial_{y_1}^j g(y_1, \eta) = -\frac{1}{c_1\eta + d_1(y_1)} \sum_{k=1}^j \binom{j}{k} d_1^{(k)}(y_1) \partial_{y_1}^{j-k} g(y_1, \eta),$$

which implies, by using the induction hypothesis, that

$$\begin{aligned} |\partial_{y_1}^j g(y_1, \eta)| &\leq B \sum_{k=1}^j \binom{j}{k} A^{k+1} (k!)^s B(AM)^{j-k+1} ((j-k)!)^s \\ &\leq B^2 (j!)^s A^{j+2} M^{j+1} \sum_{k=1}^j M^{-k}. \end{aligned}$$

In order to prove that (3.19) holds for j it suffices to show that M satisfies

$$B^2(j!)^s A^{j+2} M^{j+1} \sum_{k=1}^j M^{-k} \leq B(AM)^{j+1} (j!)^s.$$

By simplifying we obtain that the last inequality holds if and only if

$$BA \sum_{k=1}^j M^{-k} \leq 1.$$

It follows from (3.20) that the last inequality holds true. \square

Summing up we have proved that u given by (3.13) belongs to $G^s(\mathbb{T}^2)$ and it is also a matter of evaluation to show that $R_j u = f_j$, $j = 1, \dots, m$. Hence the system \mathcal{R} is globally s -solvable and therefore the proof of the sufficiency of the condition in Theorem 3.4 is now complete. \square

3.2. Global s -hypoellipticity

In this section we will study global s -hypoellipticity for the system \mathcal{R} given by (3.4).

We would like to point out that the global s -hypoellipticity on the torus \mathbb{T}^n , for the system \mathcal{R} , has been studied by [20]. As we have mentioned, our main goal in this paper is to study global s -solvability but it is known that the concepts of global s -solvability and of global s -hypoellipticity are connected and therefore it is interesting to present this connection in our situation. As we will see below when we are working on the torus \mathbb{T}^2 our statements about global s -hypoellipticity are more precise than those in the general case studied by [20]. Furthermore, our proofs are different from the ones in [20] since we take advantage of our global s -solvability results.

In contrast to what happens in the case $\Gamma = \{0\}$, here we prove that global s -hypoellipticity implies global s -solvability, but the converse is not always true.

Before we state our results we need to recall the following

Definition 3.8. We say that the system \mathcal{R} is globally s -hypoelliptic on \mathbb{T}^2 if the conditions $u \in \mathcal{D}'_s(\mathbb{T}^2)$ and $R_j u \in G^s(\mathbb{T}^2)$, $j = 1, \dots, m$, imply that $u \in G^s(\mathbb{T}^2)$.

We will need the following lemmas.

Lemma 3.9. Suppose that the system \mathcal{R} is globally s -solvable. Then \mathcal{R} is globally s -hypoelliptic on \mathbb{T}^2 if and only if $\bigcap_{j=1}^m \ker R_j \subset G^s(\mathbb{T}^2)$.

Proof. Suppose that $\bigcap_{j=1}^m \ker R_j \subset G^s(\mathbb{T}^2)$. If $u \in \mathcal{D}'_s(\mathbb{T}^2)$ is such that $R_j u = f_j \in G^s(\mathbb{T}^2)$, $j = 1, \dots, m$, then $(f_1, \dots, f_m) \in \mathcal{G}^s(\mathcal{R})$. Since \mathcal{R} is globally s -solvable, there exists $v \in G^s(\mathbb{T}^2)$ such that $R_j v = f_j$, $j = 1, \dots, m$. Hence, $u - v \in \bigcap_{j=1}^m \ker R_j$, which implies by our hypothesis that $u \in G^s(\mathbb{T}^2)$. Therefore, the system \mathcal{R} is globally s -hypoelliptic on \mathbb{T}^2 .

Conversely, if $u \in \mathcal{D}'_s(\mathbb{T}^2)$ is such that $R_j u = 0$, $j = 1, \dots, m$, then by using the fact that \mathcal{R} is globally s -hypoelliptic, it follows that $u \in G^s(\mathbb{T}^2)$. Hence, $\bigcap_{j=1}^m \ker R_j \subset G^s(\mathbb{T}^2)$. \square

We now define $J = \{1, \dots, m\}$ if $\{-c_j\eta: \eta \in \mathbb{Z}\} \cap R(d_j) \neq \emptyset$, for $j = 1, \dots, m$ and $J = \emptyset$ otherwise. When $J \neq \emptyset$, we write $\{-c_1\eta: \eta \in \mathbb{Z}\} \cap R(d_1) = \{-c_1\eta_1, \dots, -c_1\eta_N\}$ and $r_{jk}(y_1) = c_j\eta_k + d_j(y_1)$. We set $Z_k = \{y_1 \in \mathbb{T}: \sum_{j \in J} |r_{jk}(y_1)|^2 = 0\}$, with the convention that $Z_k = \emptyset$ for all k in the case $J = \emptyset$.

Lemma 3.10. *If $Z_k = \emptyset$ for $k = 1, \dots, N$, then $\bigcap_{j=1}^m \ker R_j = \{0\}$.*

Proof. We notice that

$$\begin{aligned} v \in \bigcap_{j=1}^m \ker R_j &\Leftrightarrow c_j D_{y_2} v + d_j(y_1)v = 0, \quad j = 1, \dots, m \\ &\Leftrightarrow (c_j\eta + d_j(y_1))\hat{v}(y_1, \eta) = 0, \quad j = 1, \dots, m, \quad \eta \in \mathbb{Z}. \end{aligned} \quad (3.21)$$

If $J \neq \emptyset$ we have $c_1\eta + d_1(y_1) \neq 0$ for all $y_1 \in \mathbb{T}$ and $\eta \notin \{\eta_1, \dots, \eta_N\}$. Thus $\hat{v}(y_1, \eta) = 0$ for $\eta \notin \{\eta_1, \dots, \eta_N\}$, $y_1 \in \mathbb{T}$.

When $\eta = \eta_k$ for some $k \in \{1, \dots, N\}$, we have

$$r_{jk}(y_1)\hat{v}(y_1, \eta_k) = 0, \quad j = 1, \dots, m. \quad (3.22)$$

Since $Z_k = \emptyset$, then for any $y_1 \in \mathbb{T}$ there exists $j_{y_1} \in \{1, \dots, m\}$ such that $r_{j_{y_1}k}(y_1) \neq 0$. Hence, $D_k(y_1) = \sum_{j=1}^m |r_{jk}(y_1)|^2 \neq 0$ for any $y_1 \in \mathbb{T}$.

Now, we are going to prove that $\hat{v}(y_1, \eta_k)$ is zero. It follows from (3.22) that for any $\varphi \in G^s(\mathbb{T})$ we have

$$\begin{aligned} \langle \hat{v}(y_1, \eta_k), \varphi \rangle &= \left\langle \hat{v}(y_1, \eta_k), D_k \frac{\varphi}{D_k} \right\rangle \\ &= \left\langle D_k \hat{v}(y_1, \eta_k), \frac{\varphi}{D_k} \right\rangle \\ &= \sum_{j=1}^m \left\langle |r_{jk}(y_1)|^2 \hat{v}(y_1, \eta_k), \frac{\varphi}{D_k} \right\rangle \\ &= \sum_{j=1}^m \left\langle r_{jk}(y_1) \hat{v}(y_1, \eta_k), \frac{\overline{r_{jk}(y_1)} \varphi}{D_k} \right\rangle \\ &= 0, \end{aligned}$$

and therefore $\hat{v}(y_1, \eta_k) = 0$. Thus, $v \equiv 0$, that is, $\bigcap_{j=1}^m \ker R_j = \{0\}$.

When $J = \emptyset$, there exists $j_0 \in \{1, \dots, m\}$ such that

$$\{-c_{j_0}\eta: \eta \in \mathbb{Z}\} \cap R(d_{j_0}) = \emptyset.$$

It follows from this and from (3.21) with $j = j_0$ that $\hat{v}(y_1, \eta) = 0$ for all $y_1 \in \mathbb{T}$, $\eta \in \mathbb{Z}$. Thus, $\bigcap_{j=1}^m \ker R_j = \{0\}$. \square

Finally, we have

Theorem 3.11. *Let Z_k be defined as above. The system \mathcal{R} is globally s -hypoelliptic on \mathbb{T}^2 if and only if $Z_k = \emptyset$ for all k .*

Proof. *Necessity.* Suppose that $Z_k \neq \emptyset$ for some k . Hence, $J \neq \emptyset$. Then there exists $y_1^0 \in \mathbb{T}$ such that $r_{jk}(y_1^0) = 0$ for all $j \in \{1, \dots, m\}$ and it is easy to prove that

$$R_j(\delta(y_1 - y_1^0) \otimes e^{iy_2\eta_k}) = 0, \quad j = 1, \dots, m.$$

Therefore the system \mathcal{R} is not globally s -hypoelliptic on \mathbb{T}^2 . The proof of the necessity of our condition is complete.

Sufficiency. Assume that $Z_k = \emptyset$ for all k . Notice that when d_{j_0} is non-constant for some $j_0 \in \{1, \dots, m\}$ then $\mathcal{F}_k \subset Z_k$ for all $k \in \{1, \dots, N\}$. It follows from Lemma 3.3, Theorem 3.4, Lemmas 3.9 and 3.10 that the system \mathcal{R} is globally s -hypoelliptic on \mathbb{T}^2 . \square

3.3. Original variables

We are now back to the original variables $x = (x_1, x_2) \in \mathbb{T}^2$. In this subsection we will write down our main result, Theorem 3.4, in terms of the original variables and the other ones the reader can easily write down their statements in the same fashion.

For this we recall that we are working with the \mathbb{Z} -basis of \mathbb{Z}^2 given by $\{k^1, v^2\}$.

Suppose that there exists $j_0 \in \{1, \dots, m\}$ such that $b_{j_0\Gamma}$ is non-constant and

$$\{-\langle \omega^j, v^2 \rangle \eta : \eta \in \mathbb{Z}\} \cap \mathbf{R}(b_{j\Gamma}) \neq \emptyset, \quad j = 1, \dots, m.$$

Since $\mathbf{R}(b_{j_0\Gamma})$ is a bounded set, $\{-\langle \omega^j, v^2 \rangle \eta : \eta \in \mathbb{Z}\} \cap \mathbf{R}(b_{j_0\Gamma})$ is a finite set.

Assuming that $j_0 = 1$ we set

$$\{-\langle \omega^1, v^2 \rangle \eta : \eta \in \mathbb{Z}\} \cap \mathbf{R}(b_{1\Gamma}) = \{-\langle \omega^1, v^2 \rangle \eta_1, \dots, -\langle \omega^1, v^2 \rangle \eta_N\}.$$

Let

$$R_{jk}(x) = b_{j\Gamma}(x) + c_j \eta_k, \quad j = 1, \dots, m, \quad k = 1, \dots, N.$$

In order to state our main result we shall define the following sets:

$$\mathcal{E}_{jk}^\mu = \left\{ x \in \mathbb{T}^2 : \left(\frac{\partial}{\partial \mu} \right)^\ell R_{jk}(x) = 0, \ell \geq 0 \right\}, \quad j = 1, \dots, m,$$

where

$$\mu = (v_2^2, -v_1^2)$$

and, for $\ell \geq 1$,

$$\begin{aligned} \left(\frac{\partial}{\partial \mu}\right)^\ell R_{jk}(x) &= \left(\frac{\partial}{\partial \mu}\right)^\ell b_{j\Gamma}(x) \\ &= \sum_{\substack{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2 \\ \alpha_1 + \alpha_2 = \ell}} \binom{\ell}{\alpha_1} (v_2^2)^{\alpha_1} (-v_1^2)^{\alpha_2} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} b_{j\Gamma}(x). \end{aligned}$$

We have

Theorem 3.12. *If $\dim \Gamma = 1$ then the system $Q_j = L_j + b_{j\Gamma}(x)$, $j = 1, \dots, m$, is globally s -solvable if and only if $\mathcal{E}_k^\mu \doteq \bigcap_{j=1}^m \mathcal{E}_{jk}^\mu = \emptyset$, $k = 1, \dots, N$.*

Now we are going to show that Theorem 3.12 is equivalent to Theorem 3.4.

Recall that $\mathcal{F}_k = \bigcap_{j=1}^m \{y_1 \in \mathbb{T} : r_{jk} \text{ is flat at } y_1\}$.

Now we are going to compare the sets \mathcal{F}_k and \mathcal{E}_k^μ .

By recalling that $r_{jk}(y_1) = b_{j\Gamma}(v_2^2 y_1, -v_1^2 y_1) + c_j \eta_k$ we have, for $\ell \geq 1$,

$$\begin{aligned} r_{jk}^{(\ell)}(y_1) &= \sum_{\substack{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2 \\ \alpha_1 + \alpha_2 = \ell}} \binom{\ell}{\alpha_1} (v_2^2)^{\alpha_1} (-v_1^2)^{\alpha_2} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} b_{j\Gamma}(v_2^2 y_1, -v_1^2 y_1) \\ &= \left(\frac{\partial}{\partial \mu}\right)^\ell b_{j\Gamma}(v_2^2 y_1, -v_1^2 y_1). \end{aligned}$$

Hence, if $y_1 \in \mathcal{F}_k$ then $(v_2^2 y_1, -v_1^2 y_1) \in \mathcal{E}_k^\mu$.

Conversely, if $x = (x_1, x_2) \in \mathcal{E}_k^\mu$ we take $(y_1, y_2) = M^t(x_1, x_2)$ and we will prove that $y_1 \in \mathcal{F}_k$. Indeed, by recalling that $r_{jk}(y_1) = b_{j\Gamma}(v_2^2 y_1, -v_1^2 y_1) + c_j \eta_k = b_{j\Gamma}(v_2^2 y_1 - k_2^1 y_2, -v_1^2 y_1 + k_1^1 y_2) + c_j \eta_k$ we have, for $\ell \geq 1$,

$$\begin{aligned} r_{jk}^{(\ell)}(y_1) &= \frac{d^\ell}{dy_1^\ell} b_{j\Gamma}(v_2^2 y_1 - k_2^1 y_2, -v_1^2 y_1 + k_1^1 y_2) + c_j \eta_k \\ &= \sum_{\substack{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2 \\ \alpha_1 + \alpha_2 = \ell}} \binom{\ell}{\alpha_1} (v_2^2)^{\alpha_1} (-v_1^2)^{\alpha_2} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} b_{j\Gamma}(v_2^2 y_1 - k_2^1 y_2, -v_1^2 y_1 + k_1^1 y_2) \\ &= \left(\frac{\partial}{\partial \mu}\right)^\ell b_{j\Gamma}(v_2^2 y_1 - k_2^1 y_2, -v_1^2 y_1 + k_1^1 y_2) \\ &= \left(\frac{\partial}{\partial \mu}\right)^\ell b_{j\Gamma}((M^t)^{-1}(y_1, y_2)) = \left(\frac{\partial}{\partial \mu}\right)^\ell b_{j\Gamma}(x_1, x_2). \end{aligned}$$

It follows from the formula above that if $(x_1, x_2) \in \mathcal{E}_k^\mu$ then $y_1 \in \mathcal{F}_k$.

Hence, we have proved that $\mathcal{F}_k = \emptyset$ if and only if $\mathcal{E}_k^\mu = \emptyset$.

Next we point out that our statement does not depend on the \mathbb{Z} -basis of \mathbb{Z}^2 that we are working with. In fact, one can prove the following

Remark 3.13. Let $\{\tilde{k}^1, \tilde{v}^2\}$ be another \mathbb{Z} -basis of \mathbb{Z}^2 where \tilde{k}^1 is a \mathbb{Z} -basis of Γ . Then $\mathcal{E}_k^\mu \neq \emptyset$ if and only if $\mathcal{E}_k^{\tilde{\mu}} \neq \emptyset$ where $\tilde{v}^2 = (\tilde{v}_1^2, \tilde{v}_2^2)$ and $\tilde{\mu} = (\tilde{v}_2^2, -\tilde{v}_1^2)$.

Thanks to the considerations above it follows that Theorem 3.12 does not depend on the \mathbb{Z} -basis $\{k^1, v^2\}$ of \mathbb{Z}^2 and it is equivalent to Theorem 3.4.

4. Applications

In this section we will first present an application of our results obtained in the case when $s = \infty$ and $\Gamma = \{0\}$.

Here we consider a system of real vector fields \mathcal{M} on \mathbb{T}^2 with variable coefficients given by

$$M_j = a_{j1}(x)D_{x_1} + a_{j2}(x)D_{x_2}, \quad j = 1, \dots, m, \quad (4.1)$$

where $x = (x_1, x_2) \in \mathbb{T}^2$ and the functions $a_{jk} \in C^\infty(\mathbb{T}^2; \mathbb{R})$, $j = 1, \dots, m$, $k = 1, 2$.

Now we consider functions $b_j(x) \in C^\infty(\mathbb{T}^2; \mathbb{C})$ and we will analyze the global ∞ -solvability for the following system of operators \mathcal{N} given by

$$N_j = M_j + b_j(x), \quad j = 1, \dots, m. \quad (4.2)$$

For this we will define an extension of the key set Γ when the coefficients are not constant. More precisely,

$$\Gamma' = \Gamma'(\mathcal{N}) = \{\xi \in \mathbb{Z}^2: \langle w_j(x), \xi \rangle = 0, \quad j = 1, \dots, m, \quad \forall x \in \mathbb{T}^2\}$$

and $\Gamma'^c = \mathbb{Z}^2 \setminus \Gamma'$, where $w_j(x) = (a_{j1}(x), a_{j2}(x))$, $j = 1, \dots, m$.

We can write the functions $b_j(x) = b_{j\Gamma'}(x) + b_{j\Gamma'^c}(x)$, as before.

We will need the following result:

Theorem 4.1. (See [15].) Let L be a smooth real vector field on \mathbb{T}^n such that ${}^tL: \mathcal{D}'(\mathbb{T}^n) \rightarrow \mathcal{D}'(\mathbb{T}^n)$ is globally ∞ -hypoelliptic on \mathbb{T}^n . Then there exists a unique $w = w_L \in C^\infty(\mathbb{T}^n)$ such that $w(x) > 0$, for all $x \in \mathbb{T}^n$ and $\int_{\mathbb{T}^n} w(x) dx = 1$ satisfying

$$\ker {}^tL = [w],$$

i.e., $h \in \ker {}^tL$ if and only if there exists a constant c such that $h = cw$.

Now we state the main result of this section.

Theorem 4.2. Suppose that M_j , $j = 1, \dots, m$, given by (4.1), is a family of commuting real vector fields on \mathbb{T}^2 such that tM_1 is a globally ∞ -hypoelliptic operator on \mathbb{T}^2 . Let $b_j(x) \in C^\infty(\mathbb{T}^2; \mathbb{C})$, $j = 1, \dots, m$, be functions such that $M_j b_{k\Gamma'^c} = M_k b_{j\Gamma'^c}$ and $\langle w, b_{j\Gamma'^c} \rangle = 0$ for all $j = 1, \dots, m$ where $w = w_{M_1}$.

Then we have:

1. If there exists $j_0 \in \{1, \dots, m\}$ such that $\widehat{\Im b_{j_0}}(0) \neq 0$ then \mathcal{N} is globally ∞ -solvable.

2. Suppose that $\widehat{\mathfrak{B}}_j(0) = 0$ for all $j \in \{1, \dots, m\}$. Then \mathcal{N} is globally ∞ -solvable if and only if there exist $C > 0$ and $K > 0$ such that for each $\xi \in \mathbb{Z}^2$, $|\xi| \gg 1$, there exists $j_\xi \in \{1, \dots, m\}$ such that

$$|\langle \theta^{j_\xi}, \xi \rangle + \widehat{b}_{j_\xi}(0)| \geq \frac{C}{|\xi|^K},$$

where $\theta^j = (c_{j1}, c_{j2})$ with $c_{jk} = \int_{\mathbb{T}^2} a_{jk}(x)w(x)dx$, $j = 1, \dots, m$, $k = 1, 2$.

Remark 4.3. Observe that the transpose of a vector field M is a perturbation of zero order of $-M$. In general, a perturbation of zero order of a globally hypoelliptic vector field does not preserve this property. For example, if $M = D_{x_1} + aD_{x_2}$, where a is a constant, is globally hypoelliptic then there exists c such that $-M + c$ is not (see [4]).

Remark 4.4. Even though the operators in this section have non-constant coefficients it follows from our hypotheses that we can define global ∞ -solvability for them in the same way we did before.

Proof of Theorem 4.2. First of all we will show how we can simultaneously transform our family of commuting real vector fields with variable coefficients into a family of constant vector fields. Since by hypotheses the real vector fields M_j are pairwise commuting and tM_1 is globally hypoelliptic on \mathbb{T}^2 it follows from Theorem 2.1 in [40] that there is a smooth transformation $\tau : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, $y = \tau(x)$, such that in the new variables the family $\{M_j\}_1^m$ becomes $\{X_j = \sum_{k=1}^2 c_{jk} D_{y_k}\}_1^m$, where c_{jk} are real constants. Moreover, the coefficients of X_j are given by

$$c_{jk} = \int_{\mathbb{T}^2} a_{jk}(x)w(x)dx, \quad k = 1, 2, \quad (4.3)$$

and the coefficients of X_1 , c_{1k} , satisfy the following Diophantine condition: there exist $K > 0$, $C > 0$ such that

$$\left| \sum_{k=1}^2 c_{1k} \xi_k \right| \geq \frac{C}{(1 + |\xi|)^K}, \quad \forall \xi \in \mathbb{Z}^2 \setminus \{0\}. \quad (4.4)$$

We notice that inequality (4.4) implies that the vectors $\theta^j = (c_{j1}, c_{j2})$, $j = 1, \dots, m$, are not simultaneously approximable with exponent ∞ with respect to $\mathbb{Z}^2 \setminus \{0\} = \Gamma^c(\mathcal{X})$. Thus, it follows from Theorem 2.6 that the system \mathcal{X} is globally ∞ -solvable and one can prove that it implies that the system \mathcal{M} is globally ∞ -solvable.

Now we will prove that there exists $h \in C^\infty(\mathbb{T}^2)$ such that $M_j h = b_{j\Gamma^c}$, $j = 1, \dots, m$.

For this it suffices to show that for any $u \in \bigcap_{j=1}^m \ker {}^tM_j$ we have $\langle u, b_{j\Gamma^c} \rangle = 0$ for all $j = 1, \dots, m$, since by hypothesis we have $M_j b_{k\Gamma^c} = M_k b_{j\Gamma^c}$.

Let $u \in \bigcap_{j=1}^m \ker {}^tM_j$. Thanks to the hypothesis that $\langle w, b_{j\Gamma^c} \rangle = 0$ for all $j = 1, \dots, m$ we can conclude that $\langle u, b_{j\Gamma^c} \rangle = 0$ for all $j = 1, \dots, m$, since $\bigcap_{j=1}^m \ker {}^tM_j \subset \ker {}^tM_1$ and $\ker {}^tM_1 = [w]$. Hence, we have proved that there exists $h \in C^\infty(\mathbb{T}^2)$ such that $M_j h = b_{j\Gamma^c}$, $j = 1, \dots, m$.

Now one can prove that the system \mathcal{N} is globally ∞ -solvable if and only if the system $Q_j = M_j + b_{j\Gamma'}$ is globally ∞ -solvable.

In the next step we will prove that $\Gamma' = \{0\}$ and, therefore, $b_{j\Gamma'} = \widehat{b}_j(0)$.

By using (4.3) we obtain

$$c_{11}\xi_1 + c_{12}\xi_2 = \int_{\mathbb{T}^2} (a_{11}(x)\xi_1 + a_{12}(x)\xi_2)w(x)dx, \quad \forall \xi \in \mathbb{Z}^2. \quad (4.5)$$

Suppose that there exists $\xi^0 = (\xi_1^0, \xi_2^0) \in \mathbb{Z}^2$, $\xi^0 \neq 0$, such that $\xi^0 \in \Gamma'$, i.e.,

$$a_{j1}(x)\xi_1^0 + a_{j2}(x)\xi_2^0 = 0, \quad j = 1, \dots, m \text{ and for all } x \in \mathbb{T}^2.$$

In particular, we have

$$a_{11}(x)\xi_1^0 + a_{12}(x)\xi_2^0 = 0, \quad \forall x \in \mathbb{T}^2. \quad (4.6)$$

It follows from (4.5) and (4.6) that

$$c_{11}\xi_1^0 + c_{12}\xi_2^0 = 0,$$

with $\xi^0 = (\xi_1^0, \xi_2^0) \neq 0$ which is a contradiction with (4.4). Thus, we have proved that $\Gamma' = \{0\}$.

It follows from the considerations above that the system \mathcal{N} is globally ∞ -solvable if and only if the system \mathcal{Q} given by

$$Q_j = M_j + \widehat{b}_j(0), \quad j = 1, \dots, m,$$

is globally ∞ -solvable.

By using the new variables $y = \tau(x)$, we know that the system \mathcal{Q} is globally ∞ -solvable if and only if the system \mathcal{X} given by

$$X_j + \widehat{b}_j(0), \quad j = 1, \dots, m,$$

is globally ∞ -solvable.

By using Fourier series one can easily complete the proof. \square

As our second application we consider the system \mathcal{L} of vector fields on the torus \mathbb{T}^{n+1} with variable coefficients, given by

$$L_j = \partial_{t_j} + a_j(t)\partial_x, \quad j = 1, \dots, n,$$

where $a_j \in G^s(\mathbb{T}^n; \mathbb{R})$ satisfy the following condition: $L_j a_k = \partial_{t_j} a_k = \partial_{t_k} a_j = L_k a_j$.

We define

$$h(t) = \int_0^{t_1} a_1(s, t_2, \dots, t_n) ds - a_{10}t_1 + \dots + \int_0^{t_n} a_n(0, \dots, 0, s) ds - a_{n0}t_n,$$

where $a_{j0} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} a_j(t) dt$, $j = 1, \dots, n$.

In the new variables $s_j = t_j$, $j = 1, \dots, n$, and $y = x - h(t)$ the system \mathcal{L} is simultaneously reduced to the system \mathcal{Y} of constant real vector fields given by $Y_j = \partial_{s_j} + a_{j0}\partial_y$.

Thus it follows from an n -dimensional version of Theorem 2.6 that

Theorem 4.5. *The following conditions are equivalent:*

- (I) *the vectors $\omega^1 = (1, 0, \dots, 0, a_{10})$, $\omega^2 = (0, 1, 0, \dots, 0, a_{20})$, \dots , $\omega^n = (0, \dots, 0, 1, a_{n0})$ are not SA_s ;*
- (II) *for each $(f_1, \dots, f_m) \in \mathcal{G}^s(\mathcal{L})$ there exists a unique $u \in G_{\Gamma^c}^s(\mathbb{T}^2)$ such that $L_j u = f_j$, $j = 1, \dots, m$;*
- (III) *the system \mathcal{L} is globally (s, Γ^c) -hypoelliptic on \mathbb{T}^n .*

5. Final remarks

In this section we restrict ourselves to the C^∞ case when Γ is a resonant line. The notation we use is the same one we have introduced in Section 3.

We start with the following

Lemma 5.1. *For each $\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}$ define $g_\eta \in C^\infty(\mathbb{T}; \mathbb{C})$ by*

$$g_\eta(y_1) \doteq g(y_1, \eta) = \frac{1}{c_1 \eta + d_1(y_1)}.$$

Then, for any $t \in \mathbb{R}$ there exists $C_t > 0$ such that $\|g_\eta\|_t \leq C_t$ for all $\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}$.

Proof. Since $D^j g_\eta$ is bounded uniformly in η it follows that given $M \in \mathbb{N}$ there exists C_M such that $(1 + \theta^2)^M |\widehat{g_\eta}(\theta)| \leq C_M$, for all $\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}$. The result follows immediately. \square

Now we have

Theorem 5.2. *Suppose $Z_k = \emptyset$ for all k . Given $s \in \mathbb{R}$ there exists $c_s > 0$ such that*

$$\|u\|_s \leq c_s \max_{1 \leq j \leq m} \|R_j u\|_s, \quad \text{for any } u \in C^\infty(\mathbb{T}^2). \quad (5.1)$$

Proof. We prove only when $J \neq \emptyset$ and d_1 is non-constant.

We have

$$\begin{aligned} \|u\|_s^2 &= \sum_{(\xi, \eta) \in \mathbb{Z}^2} (1 + \xi^2 + \eta^2)^s |\hat{u}(\xi, \eta)|^2 \\ &= \sum_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}} (1 + \xi^2 + \eta^2)^s |\hat{u}(\xi, \eta)|^2 \\ &\quad + \sum_{k=1}^N \sum_{\xi \in \mathbb{Z}} (1 + \xi^2 + \eta_k^2)^s |\hat{u}(\xi, \eta_k)|^2 = I + II, \quad \text{say.} \end{aligned}$$

We first estimate I . Recall that for $\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}$ we have $\hat{u}(y_1, \eta) = g_\eta(y_1) \widehat{f}_1(y_1, \eta)$. Fix some $t > |s| + 1/2$. Then, using the Cauchy–Schwarz inequality, Lemma 5.1, and the inequality

$$(1 + (\theta + \theta')^2 + \eta^2)^s \leq 2^{|s|} (1 + \theta^2)^{|s|} (1 + \theta'^2 + \eta^2)^s, \quad \forall (\theta, \theta', \eta) \in \mathbb{Z}^3,$$

we have

$$\begin{aligned} I &= \sum_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}} (1 + \xi^2 + \eta^2)^s |\hat{u}(\xi, \eta)|^2 \\ &= \sum_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}} (1 + \xi^2 + \eta^2)^s \left| \int_0^{2\pi} e^{-iy_1 \xi} g_\eta(y_1) \widehat{f}_1(y_1, \eta) dy_1 \right|^2 \\ &= \sum_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}} (1 + \xi^2 + \eta^2)^s \left| \int_0^{2\pi} e^{-iy_1 \xi} g_\eta(y_1) \sum_{\theta \in \mathbb{Z}} \widehat{f}_1(\theta, \eta) e^{iy_1 \theta} dy_1 \right|^2 \\ &= \sum_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}} (1 + \xi^2 + \eta^2)^s \left| \sum_{\theta \in \mathbb{Z}} \widehat{g}_\eta(\xi - \theta) \widehat{f}_1(\theta, \eta) \right|^2 \\ &= \sum_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}} (1 + \xi^2 + \eta^2)^s \left| \sum_{\theta \in \mathbb{Z}} \widehat{g}_\eta(\theta) \widehat{f}_1(\xi - \theta, \eta) \right|^2 \\ &\leq \sum_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z} \setminus \{\eta_1, \dots, \eta_N\}} (1 + \xi^2 + \eta^2)^s \sum_{\theta \in \mathbb{Z}} (1 + \theta^2)^t |\widehat{g}_\eta(\theta)|^2 \\ &\quad \times \sum_{\theta \in \mathbb{Z}} (1 + \theta^2)^{-t} |\widehat{f}_1(\xi - \theta, \eta)|^2 \\ &\leq C_t^2 \sum_{\xi \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z}} (1 + \xi^2 + \eta^2)^s \sum_{\theta \in \mathbb{Z}} (1 + \theta^2)^{-t} |\widehat{f}_1(\xi - \theta, \eta)|^2 \\ &= C_t^2 \sum_{\theta \in \mathbb{Z}} \sum_{\theta' \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z}} (1 + \theta^2)^{-t} (1 + (\theta + \theta')^2 + \eta^2)^s |\widehat{f}_1(\theta', \eta)|^2 \\ &\leq 2^{|s|} C_t^2 \sum_{\theta \in \mathbb{Z}} (1 + \theta^2)^{|s| - t} \sum_{\theta' \in \mathbb{Z}} \sum_{\eta \in \mathbb{Z}} (1 + \theta'^2 + \eta^2)^s |\widehat{f}_1(\theta', \eta)|^2 \\ &= c_s \|f_1\|_s^2 \leq c_s \max_{1 \leq j \leq m} \|f_j\|_s^2. \end{aligned}$$

When $\eta = \eta_k$ for some $k \in \{1, \dots, N\}$ we have that

$$\hat{u}(y_1, \eta_k) = \sum_{j=1}^m g_{jk}(y_1) \widehat{f}_j(y_1, \eta_k),$$

where $g_{jk} \in C^\infty(\mathbb{T}; \mathbb{C})$ is given by

$$g_{jk}(y_1) = \frac{\overline{r_{jk}}(y_1)}{D_k(y_1)}.$$

In the same fashion as the estimates for I , one obtains

$$II \leq c'_{s,N,m} \max_{1 \leq j \leq m} \|f_j\|_s^2.$$

The proof is now complete. \square

Remark 5.3. Since the order of R_j , $j = 1, \dots, m$, does not exceed one, the inequality in Theorem 5.2 is saying that the loss of derivatives is one (in the sense of the Sobolev norm).

We now recall a well-known fact that if an operator P , defined on \mathbb{T}^N , is globally hypoelliptic then its transpose tP is globally solvable in $\mathcal{D}'(\mathbb{T}^N)$, that is, given $f \in C^\infty(\mathbb{T}^N)$ satisfying some appropriate compatibility conditions then there exists $u \in \mathcal{D}'(\mathbb{T}^N)$ satisfying $Pu = f$. Since the hypothesis of Theorem 5.2 implies that the system R_j is globally hypoelliptic on \mathbb{T}^2 (see Theorem 3.11), then its transpose is globally solvable in $\mathcal{D}'(\mathbb{T}^2)$. Next we show that (5.1) implies that the transpose of the system R_j is globally solvable in the Sobolev spaces. More precisely, similarly as in [19], we associate to the operators R_1, \dots, R_m the following “overdetermined” system $\mathfrak{R} : C^\infty(\mathbb{T}^2) \rightarrow (C^\infty(\mathbb{T}^2))^m$ given by

$$\mathfrak{R}\varphi = (R_1\varphi, \dots, R_m\varphi) = (u_1, \dots, u_m).$$

The transpose system is the “underdetermined” system ${}^t\mathfrak{R}$ defined by

$${}^t\mathfrak{R}(u_1, \dots, u_m) = \sum_{j=1}^m {}^tR_j u_j = \varphi.$$

Let $\mathcal{E} \doteq (\bigcap_{j=1}^m \ker R_j)^\perp = (\ker \mathfrak{R})^\perp$ in $H^s(\mathbb{T}^2)$. Given $f \in H^s(\mathbb{T}^2) \cap \mathcal{E}$, we define the linear functional

$$S : \mathcal{H} \doteq \{\mathfrak{R}\varphi; \varphi \in C^\infty(\mathbb{T}^2)\} \hookrightarrow (H^{-s}(\mathbb{T}^2))^m \rightarrow \mathbb{C}$$

given by $S(\mathfrak{R}\varphi) = \langle f, \varphi \rangle$. In \mathcal{H} we consider the norm $\|(u_1, \dots, u_m)\|' \doteq \max_{1 \leq j \leq m} \|u_j\|_{-s}$.

It follows from (5.1) that S is continuous. By the Hahn–Banach theorem there exists $u = (u_1, \dots, u_m) \in (H^s(\mathbb{T}^2))^m$ such that $u(\mathfrak{R}\varphi) = \langle f, \varphi \rangle$ for any $\varphi \in C^\infty(\mathbb{T}^2)$. Hence, there exists a continuous linear operator $L : H^s(\mathbb{T}^2) \cap \mathcal{E} \rightarrow (H^s(\mathbb{T}^2))^m$ such that ${}^t\mathfrak{R}L = I$. Summarizing, given $f \in H^s(\mathbb{T}^2) \cap \mathcal{E}$ there exists $u = (u_1, \dots, u_m) = Lf \in (H^s(\mathbb{T}^2))^m$ such that ${}^t\mathfrak{R}u = \sum_{j=1}^m {}^tR_j u_j = f$. Hence, ${}^t\mathfrak{R}$ is globally solvable in H^s .

We notice that it follows from what we have just shown that if there exists $u_0 \in (H^{s_0}(\mathbb{T}^2))^m \setminus (C^\infty(\mathbb{T}^2))^m$, for some s_0 , such that $f \doteq {}^t\mathfrak{R}u_0 \in C^\infty(\mathbb{T}^2)$ then one can choose a solution to the equation ${}^t\mathfrak{R}u = f$ which is regular enough.

Another application of (5.1) concerns to global hypoellipticity. An interesting problem is to study global hypoellipticity when R_j is perturbed by pseudodifferential operators of negative

order. This has been analyzed in [20] only in the non-resonant case on \mathbb{T}^n . Another reference for this kind of problem is [41]. In the resonant case, using a standard procedure one can prove that (5.1) implies global hypoellipticity on \mathbb{T}^2 for a system of the form $P_j = R_j + a_j(y, D)$, where $a_j(y, D)$ is a pseudodifferential operator of negative order.

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